Empty Boxes – Summary

Note Title 10/10/2011

Eleven large empty boxes are placed on a table.

An unknown number of the boxes is selected and into each eight medium boxes are placed.

An unknown number of the medium boxes is selected and into each eight small boxes are placed.

At the end of this process there are 102 empty boxes.

How many boxes are there in total?

1. *Understand the problem.*

repeatedly fill 1 empty box with 8 boxes.

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At the end of this process there are 102 empty boxes.

How many boxes are there in total?

2. *Determine the goal. Determine what is given.*

Goal: final total no. of boxes

Given: initial no., final no. of empty boxes.

3. *Model the problem.*

Introduce variables t, e (natural numbers)
repeated execution of $t, e := t+8, e+7$
initially $t = e = 11$
finally $e = 102$, $t = ?$

4. *Solve the model problem.*

$8 \times e - 7 \times t$ is an invariant of $t, e := t+8, e+7$

Initially $8 \times 11 - 7 \times 11$, i.e. 11

Finally $8 \times 102 - 7 \times t = 11$, i.e. $t = 115$.

5. *Interpret the solution.*

Final total is 115 boxes.

	e	t	$8 \times e - 7 \times t$
Initially	11	11	88 - 77
	18	19	144 - 133

Finally 102

Recall $E[xs := es]$ is E after $xs := es$

e.g. $[\ x[x := x+1] = x+1\ ]$

$[\ (x+2\times y)[x,y := y,x] = y+2\times x\ ]$

E is an invariant of $xs := es$ if

$[\ E = E[xs := es]\ ]$

Example:

$[\ 8\times e - 7\times t = (8\times e - 7\times t)[t,e := t+8, e+7]\ ]$

↖ "everywhere", i.e. for all e and t

Alternative Solutions:

1. Introduce natural number n for number of times a box is filled.

Assignment: $n,e,t := n+1, e+7, t+8$

Invariants: $e - 7\times n$

$t - 8\times n$

hence $8\times(e-7\times n) - 7\times(t-8\times n)$

(i.e. $8\times e - 7\times t$).

2. (Traditional)

Introduce *functions* e, t of type $\mathbb{N} \to \mathbb{N}$.

Denote function application by subscripting.
(e.g. e_{10} is no. of empty boxes after filling 10 boxes)

$$e_0 = t_0 = 11$$

$$e_{n+1} = e_n + 7$$

$$t_{n+1} = t_n + 8$$

Calculate n when $e_n = 102$

Then calculate t_n.